Mathematical errata for "Everything and More: A Compact History of ∞", by David Foster Wallace, Atlas/Norton, 2003, 1st hardcover edition.

p. 14: Conflation of inductive reasoning – called the Principle of Induction, with an $(n+1)$ showing up – with the later, quite different, more precisely defined Principle of Mathematical Induction. Not actually wrong, but misleading.

p. 18: "Calculations involving numbers larger than 2.56×20^{92} are called transcomputational problems, meaning they're not even theoretically doable" – should be numbers requiring more than this many bits (said numbers being of size greater than 2 to the power of this large number). Typo, probably.

p. 36: "$s_n = -1$ for odd n" – should be +1.

p. 40, FN 25: "a one-to-one correspondence between their members is now actually the definition of equality between two sets" – actually, the definition of the cardinalities of two sets being equal. Sets are equal if and only if they contain exactly the same members. Repeated.

p. 69: Attepted resolution of the paradox that $0.9999\ldots = 1.0$, by saying that $0.9999.. = 1 - 1/\infty$. But in fact, $0.9999\ldots$ does equal 1.0 (as admitted on p. 79), and this is exploited later to simplify one of Cantor's proofs. This book does not treat infinitesmals as quantities you can subtract from 1, except in a later footnote about nonstandard analysis.

p. 72: The "Number Line" here includes all rationals, though this is never stated; in fact, it is introduced as if it just contains integers. Also on this page, "Every point is next to another point" as a property of the N.L makes no sense. Repeated; the book keeps trying to define "next instant", which is unnecessary (and doomed to failure).

p. 83: "What Eudoxus does is use random integers". "Random" has a very specific, probabilistic meaning in math. Should say "arbitrary".

p. 89: The argument given here, basically a natural-language version of the proof that the rationals have measure zero, makes no sense unless the reader knows that the rationals are denumerable, which is not presented for another 162 pages, in a nontrivial proof by Cantor.

p. 96: You can only derive $\log(1+x) = x - 1/2x^2 + 1/3x^3 \ldots$ if you accept that the indefinite integral of $1/(1+x)$ is $\log(1+x)$ and that term-by-term integration of infinite series is justified, none of which we should be sure of at this point.

p. 98, FN 9: "method for determining the areas/volumes of figures created by rotating curves entails treating solids as composed of n infinitesmal polygons whose areas can be summed" – confusion of area, a 2D concept, and volume, a 3D concept. If you integrate area properly, you get volume, but no one knew that at this point in history.

p. 109: "topological space" is just name-dropping without purpose.

p. 113: "if $x \geq 1$, it's easy to see that the sequence generated by expanding $1 - (1/x)$ will be so bounded" – makes no sense.

p. 115: The expression called Zeno's Dichotomy $(1/2 + 1/4 + \ldots)$ is not a power series, and neither is a Fourier series (sum of trig terms).

p. 117: "If $f(z)$ is the derivative of $f(x)$, then $f(x)$ is the integral of $f(z)$". Should read: "if $f(x)$ is the derivative of $g(x)$, then $g(x)$ is the integral of $f(x)$."

p. 123: The formal proof that $[0,1]$ can be put into 1-1 correspondence with the R.L is not more complicated than given here, and it's not done in §7 – there it's $[0,1]$ put into 1-1 correspondence with $[0,1]^2$ (the Unit Square).

p. 131: The triangle A has area $1/2xy$, not xy. Typo.

p. 132: The area doesn't increase by tz but $t(2x + z/2)$, since the increase is a trapezoid, not a rectangle. The math works out about the same.

p. 150: Why introduce partial derivatives and differentials? Only use made of them is in giving the differential equation of the Wave Function, which is unnecessary.

p. 156: The "boilerplate" skipped in the definition of uniform convergence is the important part, because a given ϵ holds for the whole interval, not point by point. This makes Cauchy's mistake incomprehensible here.

p. 157: "Some but not all monotonic series are sectionally monotonic" A sequence can be monotonic; a function can be monotonic or sectionally monotonic. Lots of confusion here. It's a straightforward consequence of the definition of "sectionally monotonic" that a sectionally monotonic function has only a finite number of discontinuities. FN 21 says we won't have to deal with monotonic functions, but that's what is necessary (via sectionally monotonic) for a function to satisfy to have a unique Fourier representation, which is why all this is done.

p. 158: "for whatever values of x make $\cos(x - \alpha) = 1$" – try $x = \pi/2 + 2\pi n + \alpha$. Definitions here of absolute and conditional convergence never used.

p. 159: d'Alembert solved the Wave Equation for $c = 1$, yielding a solution $y = f(x+t) + g(x-t)$, which Euler generalized. This has an intuitive explanation: if you pluck a string and let it go, you can think of two "waves" travelling to left and right from the point you plucked. d'Alembert's solution does not work only if the initial curve is a periodic function; his solution worked for a curve pinned down to 0 at $x = 0$ and $x = 1$. Daniel Bernoulli came up with another solution, in terms of trig series, for which the solution was a periodic function. The exploration of which solution was more general prompted the whole study of Fourier series.

p. 166, FN 40: Product of $\sqrt{2}$ and $\sqrt{3}$ is $\sqrt{6}$, not their sum.

p. 174, FN 47: Lots of confusion. $\sum a_n$ is convergent (not uniformly convergent, a concept which applies to series of functions, not of numbers). This makes a hash of talk here of circularity removed by "simpler parts", as in induction. Also the $\sum c_n$ should be $\sum c_n(x)$.

p. 183, FN 64: Fourier series are not power series.

p. 190: Use of Weierstrass's extreme values theorem to find the "next instant" seems to be gibberish. If time is a function, a function of what? What is the dependent variable and what is the independent variable?

p. 191 From here to halfway through page 195, the attempt to give us an epsilon-delta answer to Zeno's Dichotomy, is pretty much completely horizontally-mamboed. Details provided below, on page 3.

p. 206, FN 10: Fractions don't have to be reduced to lowest terms, though it doesn't hurt.

p. 207, FN 12: The word *class* is now used to refer to anything that can be formed by Unrestricted Abstraction, some of which are not sets (hence the need for Restricted Abstraction). Cantor et al. did not make this distinction.

p. 220: Probably should say "defines a real number if, for any m, $\lim_{n->\infty}(a_{n+m} - a_n) = 0$", and as FN says, probably shouldn't use limits at all here, but an epsilon rule – incorrect in FN 31, which should read, "by the rule that for any m and any ϵ, there exists an n_0 such that for all $n > n_0, a_{n+m} - a_n < \epsilon$".

p. 222: "the same way that 0.15 is Lim $(0(10^0), 1(10^1), 5(10^2))$". $0.15 = 0 \times 10^0 + 1 \times 10^1 + 5 \times 10^2$, no need to invoke a limit, notation makes no sense. Later on: "when a fundamental sequence $a_0, \ldots, a_n, \ldots$ has each of its terms after a_n equal to either 0 or a, the sequence defines (=is) the rational number a". 0 should be left out of this; the sequence $0, a, 0, a, 0, a \ldots$ does not define a, as it doesn't converge by the definition above.

p. 231, FN 11: Any trig series is integrable term by term, but the result might not converge.

p. 233: "if any derived set P^n is finite, then at some further point $n + k$ the derived set $P^{(n+k)}$ is going to take its absolute minimum value m, which in this case will be 0". If P^n is finite, P^{n+1} will be empty (not zero) as needed. Sets don't "take" values.

p. 235, FN 16: This is not quite what decidability is in logic. Decidability is the property of a formal system that says that there is a procedure to decide if any given logical expression is true or false. You can speak of a set being decidable if there is a decision procedure for membership, but that idea comes out of the work of Alan Turing on computability.

p. 241: Not clear what is being done by induction. Also, FN 27 is pretty much nonsense.

p. 248: You can define denumerable as a 1-1 correspondence, but it's more common to define it as an enumeration $a_0, a_1, \ldots$ such that $A = \{a_0, a_1, \ldots\}$. The crucial difference is that this definition allows repetitions, ie a_1 could equal a_2. This weaker definition is equivalent to the stronger one, but easier to work with; on p. 252, the elimination of unreduced fractions wouldn't be necessary with the weaker definition.

p. 253: "one of the most important proof-techniques in all of number theory" should say "set theory".

p. 254, FN 49: The Axiom of Choice isn't needed for this proof; the 1-1 correspondence (which we're trying to prove by *reductio* doesn't exist) gives the first number in the set, namely the one that 1 maps to.

p. 266, FN 65: Should be $P(\bar{A}) = 2^{\bar{A}}$, though 2^A isn't defined for infinite sets yet. General confusion between sets and their cardinalities in this section.

p. 267: Proof here only works for finite A (even the rigourous proof by induction). Also true for the intuition on p. 270.

p. 268: "$P(A) > A$" really should be "$P(A) \supsetneq A$", but it's okay to overload ">" here – still, it should be defined. More confusion between sets and cardinalities.

p. 273, FN 70: Editor partly right here; this is a constructive proof, though not in Kronecker's sense (because the construction goes on forever). It's perfectly all right to regard this as "Given a 1-1 correspondence between I and a set of subsets of I, we can construct a subset of I onto which no element of I maps."

p. 283: "Gödel's aforementioned proofs that a formal system can't be both complete and consistent" (cf. New Yorker review). Both propositional and predicate logic are complete and consistent (latter Gödel's PhD thesis). It should say "a formal system that is powerful enough to express number-theoretic statements is either inconsistent or incomplete, and if it is consistent, its consistency cannot be proved within the system."

p. 286: Transfinite induction, to be different from regular induction, has to include forming a limit set (e.g. the infinite union of $\phi, \{\phi\}, \{\{\phi\}\}, \ldots$), and arguing that if every element of a limit set satisfies some property, then the limit set does. What's stated here is just regular induction.

p. 289: The well-ordering principle isn't necessary for Cantor's proofs, for the same reason as on p. 254.

p. 290: The Axiom of Regularity doesn't break Cantor's second proof of "$P(A) > A$". It breaks Russell's Antimony which is inspired by that proof.

p. 296: None of the $\forall$s should be here. Should say $n = \{x \mid x < n\}$, and so on.

P. Ragde, with some material from J. Ellenberg

Fix to page 191-195:

The problem is the given definition of the limit is that of a function approaching a point on the R.L. But what is needed is the definition of the limit of a set of finite sums (each one the partial sum of an infinite series) as the number of things summed goes to infinity. The book more or less says this at the top of page 193. Trying to use the wrong definition here (the right one is very similar, but different in crucial ways) is what pretty much nullifies this section.

"Here L and x_n are both 1, $f(x)$ is $(1 - 1/2^n)$, and x can be whatever we want – the simplest way to do the proof is to let $x =$ the point $1/2^n$." That definition of $f(x)$ makes no sense, unless we substitute the definition of x into the definition of $f(x)$ and decide that the problem is to prove that $\lim_{x\to 0} 1 - x = 1$. You don't need a high-powered definition of limit to do this; you can just let x reach 0, and find that the limit is 1.

The series used here is derived from Zeno's Dichotomy, and it's $1/2+1/4+1/8+\ldots+1/2^k+\ldots$, which we can denote as $\sum_{k=1}^{\infty} 1/2^k$. What DFW wants to do is show formally that this sum is 1, that is, we can cross the street.

He sort of sneakily says that the partial sum $s_n = \sum_{k=1}^{n} 1/2^k$ satisfies $s_n = 1 - 1/2^n$. This is true, but it requires proof.

Theorem: For $n > 0$, $\sum_{k=1}^{n} 1/2^k = 1 - 1/2^n$.

Proof: By induction on n.

Base case: $n = 1$. $\sum_{k=1}^{1} 1/2^k = 1/2 = 1 - 1/2^1$, as required.

Inductive step: Assume the statement in the theorem true for $n = j$, and prove it for $n = j+1$.

For $n = j$, we have $\sum_{k=1}^{j} 1/2^k = 1 - 1/2^j$ by assumption.

For $n = j + 1$, we have

$$\begin{aligned}\sum_{k=1}^{j+1} 1/2^k &= 1/2^{j+1} + \sum_{k=1}^{j} 1/2^k \\ &= 1/2^{j+1} + 1 - 1/2^j \text{ by assumption} \\ &= 1 - 1/2^j + 1/2^{j+1} = 1 - 1/2^{j+1}\end{aligned}$$

which proves the theorem for $n = j + 1$.

Thus, by the principle of mathematical induction, the statement is true for all n.

End of proof of theorem.

We now know that $s_n = 1 - 1/2^n$, and we want to show that $\lim_{n\to\infty} s_n = 1$.

This requires a definition of limit as something approaches infinity. This looks like the definition at the bottom of p. 186, but it replaces the δ in $|x - x_n| < \delta$ (implying x is getting close to x_n) with something (which we'll call n_0) that says n is getting large (namely that it is bigger than n_0). The reason for this change is that the definition of limit DFW gives talks about x approaching a fixed value, which means it's getting closer and closer to it, as opposed to n approaching infinity, meaning it is getting larger and larger. But the idea is the same. Here's the full definition:

$\lim_{n\to\infty} s_n = L$ if for any ϵ, there exists an n_0 such that for all $n > n_0$, $|s_n - L| < \epsilon$.

Now, let's use this with our particular s_n to show that the limit of $s_n = 1 - 1/2^n$ as n approaches infinity is 1 (=L). We'll use the game on page 185. You pick ϵ, and I find n_0 such that for any $n > n_0, |s_n - L| < \epsilon$.

So you pick some ϵ. And I pick n_0 such that $1/2^{n_0} < \epsilon$. If your ϵ is 1/100, I pick $n_0 = 7$, because $1/2^7 = 1/128 < 1/100$. (This is what DFW is trying to do in the middle of page 194.) Pretty clearly, for any epsilon you choose, I can choose an n_0. (Formally, I can choose n_0 to be the smallest integer greater than the negative of the logarithm to the base 2 of ϵ. You probably didn't want to know that.)

Now, why does my n_0 work? Because if $n > n_0$, we can reason that $|s_n - 1| = |(1 - 1/2^n) - 1| = 1/2^n < 1/2^{n_0} < \epsilon$, that is, $|s_n - 1| < \epsilon$, as required.

Thus $\lim_{n\to\infty} s_n = 1$, and we can cross the street. (To show we can do it in a finite time, we had better sum up the time it takes to get across – but this is the same sum, really.)